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Abstract. We show that a del Pezzo fibration $\pi : V \to W$ of degre d contains a vertical open cylinder, that is, an open subset whose intersection with the generic fiber of π is isomorphic to $Z \times \mathbb{A}^1_K$ for some quasi-projective variety Z defined over the function field K of W, if and only if $d \geq 5$ and $\pi : V \to W$ admits a rational section. We also construct twisted cylinders in total spaces of threefold del Pezzo fibrations $\pi : V \to \mathbb{P}^1$ of degree $d \leq 4$.

Introduction

An $\mathbb{A}^r_k$-*cylinder* in a normal algebraic variety X defined over a field k is a Zariski open subset U isomorphic to $Z \times \mathbb{A}^r_k$ for some algebraic variety Z defined over k. Complex projective varieties containing cylinders have recently started to receive a lot of attention in connection with the study of unipotent group actions on affine varieties. Namely, it was established in [7, 8] that the existence of a nontrivial action of the additive group $\mathbb{G}_{a,\mathbb{C}}$ on the affine cone associated with a polarized projective variety (X, H) is equivalent to the existence in X of a so-called H-polar cylinder, that is, an $\mathbb{A}^1$-cylinder $U = X \setminus \mathrm{Supp}(D)$ for some effective divisor $D \in |mH|$, $m \geq 1$.

Since a complex projective variety X containing a cylinder is in particular birationally ruled, if it exists, the output V of a Minimal Model Program run on X is necessarily is a Mori fiber space, that is, a projective variety with $\mathbb{Q}$-factorial terminal singularities equipped with an extremal contraction $\pi : V \to W$ over a lower dimensional normal projective variety W. In case where $\dim W = 0$, V is a Fano variety of Picard number one. If $\dim V = 2$ then V isomorphic to $\mathbb{P}^2$ hence contains many (anti-canonically polar) cylinders. In higher dimension, several families of examples of Fano varieties of dimension 3 and 4 and Picard number one admitting (anti-canonically polar) cylinders have been constructed [10, 15, 16], but a complete classification is still far from being known.

In this article, we consider the question of existence of cylinders in other possible outputs of Minimal Model Programs: del Pezzo fibrations $\pi : V \to W$, which correspond to the case where $\dim W = \dim V - 2$. The general closed fibers of such fibrations are smooth del Pezzo surfaces, the degree $\deg(V/W)$ of the del Pezzo fibration $\pi : V \to W$ is then defined as the degree of such a general fiber. Del Pezzo surfaces contain many cylinders, indeed being isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ in at most eight points in general position, they even have the property that every of their closed points admits an open neighborhood isomorphic to $\mathbb{A}^2$. One could therefore expect that given a del Pezzo fibration $\pi : V \to W$, some suitably chosen families of "fiber wise" cylinders can be arranged into a "relative" cylinder with respect to $\pi : V \to W$, more precisely a so-called *vertical cylinder*:

Definition. Let $f : X \to Y$ be a morphism between normal algebraic varieties defined over a field k and let $U \simeq Z \times \mathbb{A}^r_k$ be an $\mathbb{A}^r_k$-cylinder inside X. We say that U is *vertical with respect to* f if the restriction $f|_U$ factors as

$$f\,|_U = h \circ \mathrm{pr}_Z : U \simeq Z \times \mathbb{A}^r_k \xrightarrow{\mathrm{pr}_Z} Z \xrightarrow{h} Y$$

for a suitable morphism $h : Z \to Y$. Otherwise, we say that U is a *twisted* $\mathbb{A}^r_k$-*cylinder* with respect to f.

For dominant morphisms $f : X \to Y$ the existence inside X of an $\mathbb{A}^r_k$-cylinder vertical with respect to f translates equivalently into that of an $\mathbb{A}^r_K$-cylinder inside the fiber X_η of f over the generic point η of Y, considered as a variety defined over the function field K of Y (see Lemma 3 below). It follows in particular

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that, for instance, a del Pezzo fibration $\pi : V \to W$ of degree 9 whose generic fiber V_η is a Severi-Brauer surface without rational point over the function field of W cannot contain any vertical $\mathbb{A}^1_{\mathbb{C}}$-cylinder. Our first main result is a complete characterization of del Pezzo fibrations admitting vertical cylinders:

Theorem 1. *A del Pezzo fibration $\pi : V \to W$ admits a vertical $\mathbb{A}^1_{\mathbb{C}}$-cylinder if and only if $\deg(V/W) \geq 5$ and $\pi : V \to W$ has a rational section.*

It follows in particular from this characterization that a del Pezzo fibration fibration $\pi : V \to \mathbb{P}^1$ of degree $d \leq 4$ does not admit any vertical $\mathbb{A}^1_{\mathbb{C}}$-cylinder. But the reasons which prevent the existence of such vertical cylinders does not give much insight concerning that of twisted cylinders in total spaces of such fibrations. By general results due to Alekseev [1] and Pukhlikov [17], "most" del Pezzo fibrations $\pi : V \to \mathbb{P}^1$ of degree $d \leq 4$ with smooth total spaces are non-rational. On the other hand, since $\pi : V \to \mathbb{P}^1$ has a section by virtue of the Tsen-Lang Theorem, it follows from [13, Theorem 29.4] that the total space of such a fibration of degree $d \geq 3$ is always unirational. As a consequence, "most" del Pezzo fibrations $\pi : V \to \mathbb{P}^1$ of degree $d = 3, 4$ with smooth total spaces cannot contain any $\mathbb{A}^1_{\mathbb{C}}$-cylinder at all, vertical or twisted. Indeed if V contains an open subset $U \simeq Z \times \mathbb{A}^1_{\mathbb{C}}$ for some smooth quasi-projective surface Z, then Z is unirational hence rational, which would imply in turn the rationality of V.

Our second main result consists of a construction of families of del Pezzo fibrations $\pi : V \to \mathbb{P}^1$ of any degree $d \leq 4$ containing twisted cylinders of maximal possible dimension, which arise as projective completions of $\mathbb{A}^3_{\mathbb{C}}$:

Theorem 2. *For every $d \leq 4$, there exist del Pezzo fibrations $\pi : V \to \mathbb{P}^1$ of degree d whose total spaces contain $\mathbb{A}^3_{\mathbb{C}}$ as a twisted cylinder.*

The scheme of the article is as follows. The first section is devoted to the proof of Theorem 1: we first establish in subsection 1.2 that a del Pezzo fibration of degree $d \leq 4$ does not admit vertical $\mathbb{A}^1_{\mathbb{C}}$-cylinder, then in subsection 1.3 we give explicit constructions of $\mathbb{A}^1$-cylinders inside generic fibers of del Pezzo fibrations $\pi : V \to W$ of degree $d \geq 5$ with a rational section. We also discuss as a complement in subsection 1.4 the existence of vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinders in del Pezzo fibrations $\pi : V \to W$ of degree $d \geq 5$. Then in section two, we first review the general setup for the construction of projective completions of $\mathbb{A}^3_{\mathbb{C}}$ into total spaces of del Pezzo fibrations $\pi : V \to \mathbb{P}^1$ established in [4]. Then we proceed in detail to the construction of such completions for the specific case $d = 4$, which was announced without proof in [4].

1. Vertical cylinders in del Pezzo fibrations

Letting $\pi : V \to W$ be a del Pezzo fibration over a normal projective variety W with function field K, the following Lemma 3 implies that the existence of an $\mathbb{A}^1$-cylinder $U \subset V$ vertical with respect to π is equivalent to that of an $\mathbb{A}^1_K$-cylinder in the fiber S of π over the generic point of W. On the other hand, the existence of a rational section of $\pi : V \to W$ is equivalent to the existence of a K-rational point of S. Since S is a smooth del Pezzo surface defined over K, of degree $\deg(V/W)$ and with Picard number $\rho_K(S)$ equal to one, to establish Theorem 1, it is thus enough to show that such a smooth del Pezzo surface admits an $\mathbb{A}^1_K$-cylinder if and only if it has degree $d \geq 5$ and a K-rational point.

Lemma 3. *Let $f : X \to Y$ be a dominant morphism between normal algebraic varieties over a field k. Then X contains a vertical $\mathbb{A}^r_k$-cylinder with respect to f if and only if the generic fiber of f contains an open subset of the form $T \times \mathbb{A}^r_K$ for some algebraic variety T defined over the function field K of Y.*

Proof. Indeed, if $Z \times \mathbb{A}^r_k \simeq U \subset X$ is a vertical $\mathbb{A}^r_k$-cylinder with respect to f then

$$U \times_Y \mathrm{Spec}(K) \simeq (Z \times \mathbb{A}^r_k) \times_Y \mathrm{Spec}(K) \simeq (Z \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)) \times \mathbb{A}^r_K$$

is an $\mathbb{A}^r_K$-cylinder contained in the fiber X_η of f over the generic point η of Y. Conversely, if $V \simeq T \times \mathbb{A}^r_K$ is an $\mathbb{A}^r_K$-cylinder inside X_η, then let Δ be the closure of $X_\eta \setminus V$ in X and let $X_0 = X \setminus \Delta$. The projection $\mathrm{pr}_T : V \to T$ induces a rational map $\rho : X_0 \dashrightarrow \overline{T}$ to a projective model $\overline{T}$ of the closure of T in X, whose generic fiber is isomorphic to the affine space $\mathbb{A}^r_{K'}$ over the function field K' of T. It follows that there exists an open subset $\overline{T}_0$ of $\overline{T}$ over which ρ is regular such that $\rho^{-1}(\overline{T}_0) \simeq \overline{T}_0 \times \mathbb{A}^r_k$. Replacing $\overline{T}_0$ if necessary by a smaller open subset Z on which the rational map $\overline{T} \dashrightarrow Y$ induced by f is regular, we obtain an $\mathbb{A}^r_k$-cylinder $U = \rho^{-1}(Z)$ in X which is vertical with respect to f. □

Before we proceed to the proof of Theorem 1 in the next three subsections, we list some corollaries of it.

Corollary 4. *Let $\pi : V \to C$ be a complex del Pezzo fibration over a curve C. If $\deg(V/C) \geq 5$ then V contains a vertical $\mathbb{A}^1$-cylinder.*

Proof. Indeed, the generic fiber of π is then a smooth del Pezzo surface over the function field K of C, and hence it has a K-rational point by virtue of the Tsen-Lang Theorem (see e.g. [5, Theorem 3.12]). ☐

Corollary 5. *Every complex del Pezzo fibration $\pi : V \to W$ of degree $\deg(V/W) = 5$ contains a vertical $\mathbb{A}^1$-cylinder.*

Proof. Indeed, by virtue of [19], every del Pezzo surface of degree 5 over a field K has a K-rational point. ☐

1.1. **Rationality of del Pezzo surfaces with Picard number one containing a cylinder.** Our first observation is that the rationality of a del Pezzo surface S of Picard number one is a necessary condition for the existence of an $\mathbb{A}^1$-cylinder in it.

Proposition 6. *Let S be a smooth del Pezzo surface defined over a field K of characteristic zero 0 with Picard number one. If S contains an $\mathbb{A}^1_K$ cylinder $U \simeq Z \times \mathbb{A}^1_K$ over a smooth curve Z defined over K, then S is rational.*

Given an $\mathbb{A}^1_K$-cylinder $U \simeq Z \times \mathbb{A}^1_K$ inside S, the projection $\mathrm{pr}_Z : U \to Z$ induces a rational map $\varphi : S \dashrightarrow \overline{Z}$ over the smooth projective model $\overline{Z}$ of Z. Note that S is rational provided that $\overline{Z}$ is rational, hence isomorphic to $\mathbb{P}^1_K$. The assertion of the proposition is thus a direct consequence of the following lemma:

Lemma 7. *Under the assumption of Proposition 6, the following hold:*

a) The rational map $\varphi : S \dashrightarrow \overline{Z}$ is not regular and it as a unique proper base point, which is K-rational.
b) The curve $\overline{Z}$ is rational.

Proof. Since $\mathrm{Pic}(S) \simeq \mathbb{Z}$, φ is strictly rational. Indeed, otherwise it would be a $\mathbb{P}^1$-fibration admitting a section H defined over K and the natural homomorphism $\mathrm{Pic}(\overline{Z}) \oplus \mathbb{Z}\langle H\rangle \to \mathrm{Pic}(S)$ would be injective. The rational map $\varphi_{\overline{K}} : S_{\overline{K}} \dashrightarrow \overline{Z}_{\overline{K}}$ obtained by base change to the algebraic closure $\overline{K}$ of K is again strictly rational, extending the $\mathbb{A}^1$-fibration $\mathrm{pr}_{Z_{\overline{K}}} : U_{\overline{K}} \simeq Z_{\overline{K}} \times \mathbb{A}^1_{\overline{K}} \to Z_{\overline{K}}$. Since the closed fibers of $\mathrm{pr}_{Z_{\overline{K}}}$ are isomorphic to $\mathbb{A}^1_{\overline{K}}$, a general member of $\varphi_{\overline{K}}$ is a projective curve with a unique place at infinity. It follows that $\varphi_{\overline{K}}$ has a unique proper base point. This implies in turn that $\varphi : S \dashrightarrow \overline{Z}$ has a unique proper base point p, which is necessarily K-rational. The same argument implies that a minimal resolution $\sigma : \tilde{S} \to S$ of the indeterminacies of φ is obtained by blowing-up a finite sequence of K-rational points, the last exceptional divisor produced being a section $H \simeq \mathbb{P}^1_K$ of the resulting $\mathbb{P}^1$-fibration $\tilde{\varphi} = \varphi \circ \sigma : \tilde{S} \to \overline{Z}$. This implies that $\overline{Z} \simeq \mathbb{P}^1_K$. ☐

Remark 8. In the previous lemma, the hypothesis that the Picard number of S is equal to one is crucial to infer that the projection morphism $\mathrm{pr}_Z : U \simeq Z \times \mathbb{A}^1_K \to Z$ cannot extend to an everywhere defined $\mathbb{P}^1$-fibration $\varphi : S \to \overline{Z}$. Indeed, for instance, letting $\overline{Z}$ be a smooth geometrically rational curve without K-rational point, $S = \overline{Z} \times \mathrm{Proj}_K(K[x,y])$ is a smooth del Pezzo surface of Picard number $\rho_K(S) = 2$ and degree 8, without K-rational point, hence in particular non K-rational, containing a cylinder $U = S \setminus (\overline{Z} \times [0:1]) \simeq \overline{Z} \times \mathbb{A}^1_K$.

1.2. **Non-existence of $\mathbb{A}^1$-cylinders in del Pezzo surfaces of degree $d \leq 4$.** Recall [13, 29.4.4., (iii), p.159] that the degree d of a smooth del Pezzo surface S of Picard number one ranges over the set $\{1,2,3,4,5,6,8,9\}$. By classical results of Segre-Manin [18, 12] and Iskovskikh [6], such a surface of degree $d \leq 4$ is not rational over K. Combined with Proposition 6, this implies the following:

Proposition 9. *A smooth del Pezzo surface defined over a field K of characteristic zero 0 of degree $d \leq 4$ and with Picard number one does not contain any $\mathbb{A}^1_K$-cylinder.*

Proof. We find enlightening to give an alternative argument which does not explicitly rely on the aforementioned results of Segre-Manin-Iskovskikh. So suppose for contradiction that S contains an open subset $U \simeq Z \times \mathbb{A}^1_K$ for a certain smooth curve Z defined over K. By Lemma 7, the rational map $\varphi : S \dashrightarrow \overline{Z}$ to the smooth projective model $\overline{Z} \simeq \mathbb{P}^1_K$ of Z induced by the projection $\mathrm{pr}_Z : S \to Z$ has a unique proper base point p, which is a K-rational point of S. Let $\sigma : \tilde{S} \to S$ be a minimal resolution of the indeterminacies of φ, consisting of a finite sequence of blow-ups of K-rational points, with successive exceptional divisors $E_i \simeq \mathbb{P}^1_K$, $i = 1, \ldots, n$, the last one being a section of the resulting $\mathbb{P}^1$-fibration $\tilde{\varphi} = \varphi \circ \sigma : \tilde{S} \to \overline{Z}$, and $\mathcal{L}$ be the mobile linear system on S corresponding to $\varphi : S \dashrightarrow \overline{Z}$. Since $\mathrm{Pic}(S)$ is generated by $-K_S$, it follows that

$\mathcal{L}$ is contained in the complete linear system $|-\mu K_S|$ for some positive integer μ. Letting $\tilde{\mathcal{L}}$ be the proper transform of $\mathcal{L}$ on $\tilde{S}$, we have

$$K_{\tilde{S}} + \frac{1}{\mu}\tilde{\mathcal{L}} = \sigma^*(K_S + \frac{1}{\mu}\mathcal{L}) + \sum_{i=1}^{n} \alpha_i E_i$$

for some rational numbers α_i. Since E_n is a section of $\tilde{\varphi}$ while the divisors E_i, $i = 1, \ldots, n-1$ are contained in fibers of $\tilde{\varphi}$, taking the intersection with a general K-rational fiber $F \simeq \mathbb{P}^1_K$ of $\tilde{\varphi}$, we obtain:

$$-2 = K_{\tilde{S}} \cdot F = (K_{\tilde{S}} + \frac{1}{\mu}\tilde{\mathcal{L}}) \cdot F = \sigma^*(K_S + \frac{1}{\mu}\mathcal{L}) \cdot F + \sum_{i=1}^{n} \alpha_i E_i \cdot F = \alpha_n$$

as $K_S + \frac{1}{\mu}\mathcal{L} \sim_{\mathbb{Q}} 0$ by the choice of μ. The pair $(S, \frac{1}{\mu}\mathcal{L})$ is thus not log-canonical at p. By [2, Theorem 3.1 p. 275], the local intersection multiplicity $\mathcal{L}_p^2$ at p of two members of $\mathcal{L}$ over general K-rational points of $\overline{Z}$ then satisfies $\mathcal{L}_p^2 > 4\mu^2$. But on the other hand, since p is the unique proper base point of φ, we have $\mathcal{L}_p^2 = \mathcal{L}^2 = (-\mu K_S)^2 = d\mu^2$, a contradiction since $d \leq 4$ by hypothesis. □

Remark 10. The main ingredient of the proof of Proposition 9 lies in an application of Corti's version of Pukhlikov's $4n^2$ inequality to the pair $(S, \frac{1}{\mu}\mathcal{L})$, which is not log-canonical at p. We could have also infered the same result for the case $d \leqq 3$ from the fact that any smooth del Pezzo surface X of degree $d \leqq 3$ defined over an algebraically closed field of characteristic zero does not have an anti-canonically polar cylinder [3, 9]. Indeed, since the Picard group of S is generated by $-K_S$, every $\mathbb{A}^1_K$-cylinder $U \subset S$ is automatically anti-canonically polar. The existence of such a cylinder would imply in turn that the base extension $X = S_{\overline{K}}$ of S to the algebraic closure $\overline{K}$ of K has an anti-canonically polar $\mathbb{A}^1_{\overline{K}}$-cylinder in contradiction to the aforementioned fact. Note that this argument is no longer applicable to the case $d = 4$, as every smooth del Pezzo surface defined over an algebraically closed field of characteristic zero does contain anti-canonically polar cylinders as explained in Example 11 below.

Example 11. Every smooth del Pezzo surface X of degree 4 defined over an algebraically closed field of characteristic zero is isomorphic to the blow-up $h : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ at five points in general position, say $p_1, \cdots, p_5$. Let C be the smooth conic passing through all of p_i's and let ℓ be the tangent line to C at a point q distinct from the p_i's. For a rational number $\epsilon > 0$, we have

$$-K_S \sim_{\mathbb{Q}} (1+\epsilon)C' + (1-2\epsilon)\ell' + \epsilon \sum_{i=1}^{5} E_i,$$

where C' and ℓ' are the respective proper transforms of C and ℓ, and $E_i := h^{-1}(p_i)$ $(1 \leqq i \leqq 5)$. For sufficiently small ϵ, the right hand side of the above expression is an effective divisor and on the other hand, we have:

$$S \backslash (C' \cup \ell' \cup E_1 \cup \cdots \cup E_5) \cong \mathbb{P}^2 \backslash (C \cup \ell) \cong (\mathbb{A}^1_*) \times \mathbb{A}^1.$$

It follows that S has a $(-K_S)$-polar $\mathbb{A}^1$-cylinder, actually a one-dimensional family of such cylinders parametrized by the points $q \in C \setminus \{p_1, \ldots, p_5\}$.

1.3. **$\mathbb{A}^1$-cylinders in del Pezzo surfaces of degree $d \geq 5$.** By virtue of [13, Theorem 29.4], a smooth del Pezzo surface of degree $d \geq 5$ and arbitrary Picard number is rational if and only if it admits a K-rational point. On the other hand it follows from Proposition 6 that for such surfaces with Picard number one, rationality is a necessary condition for the existence of an $\mathbb{A}^1_K$-cylinder. The following result implies in particular these two properties are in fact equivalent:

Proposition 12. *A smooth del Pezzo surface S defined over a field K of characteristic zero* 0, *of degree $d \geq 5$ and with a K-rational point admits an $\mathbb{A}^1_K$-cylinder.*

Proof. We only treat the case where S has Picard number one, the other cases are similar and left to the reader. Building on the proof of [13, Theorem 29.4], we exhibit below an explicit $\mathbb{A}^1_K$-cylinder in S for each $d = 9$, 8, 6 and 5 respectively.

(i) If $d = 9$ then by [13, 29.4.4 (i)], S is isomorphic over K to $\mathbb{P}^2_K$ and we obtain an $\mathbb{A}^1_K$-cylinder $U \simeq \mathbb{A}^2_K \simeq \mathbb{A}^1_K \times \mathbb{A}^1_K$ by taking the complement of a line $L \simeq \mathbb{P}^1_K$.

(ii) If $d = 8$, then since $\rho(S) = 1$, it follows from [13, 29.4.4 (ii)] that S is isomorphic to a smooth quadric in $\mathbb{P}^3_K$ which is a nontrivial K-form of $\mathbb{P}^1_K \times \mathbb{P}^1_K$, i.e., S is not isomorphic to $\mathbb{P}^1_K \times \mathbb{P}^1_K$ but the base extension

$S_{\overline{K}}$ of S to the algebraic closure $\overline{K}$ of K is isomorphic to $\mathbb{P}^1_{\overline{K}}\times\mathbb{P}^1_{\overline{K}}$. Given a K-rational point p of S, the union in $S_{\overline{K}}\simeq\mathbb{P}^1_{\overline{K}}\times\mathbb{P}^1_{\overline{K}}$ of the fibers of the first and second projections passing through p is a curve C defined over K. Letting $g: S'\to S$ be the blow-up of S at p, with exceptional divisor $E\simeq\mathbb{P}^1_K$, the proper transform of C in $S'_{\overline{K}}$ is a pair of disjoint (-1)-curves whose union is defined over K. The composition $\varphi: S \dashrightarrow \mathbb{P}^2_K$ of g^{-1} with the contraction of the proper transform of C is thus a rational map defined over K, mapping E to a line and restricting to an isomorphism between $U=S\setminus C\simeq S'\setminus(C\cup E)$ and the complement of the image of E in $\mathbb{P}^2_K$, which isomorphic to $\mathbb{A}^2_K$.

(iii) If $d=6$, then $S_{\overline{K}}$ is isomorphic to the blow-up $\tau: S_{\overline{K}}\to\mathbb{P}^2_{\overline{K}}$ of $\mathbb{P}^2_{\overline{K}}$ at three points x_1, x_2, x_3 in general position. The union D of all the (-1)-curves in $S_{\overline{K}}$ is defined over K, and consists of the exceptional divisors of τ and the proper transforms of the lines in $\mathbb{P}^2_{\overline{K}}$ passing through x_i and x_j, $1\le i<j\le 3$. Since $\rho(S)=1$ it follows from [13, 29.4.4 (iv)] that a K-rational point p of S is necessarily supported outside of D. Let $g: S'\to S$ be the blow-up of a K-rational point p, with exceptional divisor $E_0\simeq\mathbb{P}^1_K$. Then S' is a del Pezzo surface of degree 5, and the union D' of all (-1)-curves in $S'_{\overline{K}}$ consists of $E_{0,\overline{K}}$, the proper transform of D and the proper transforms ℓ_i by $\tau\circ g_{\overline{K}}$ of the lines in $\mathbb{P}^2_{\overline{K}}$ passing through $\tau(p)$ and x_i, $i=1,\ldots,3$. By construction, $E_{0,\overline{K}}$ is invariant under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$ on $S'_{\overline{K}}$ and ℓ_1, ℓ_2, ℓ_3 are the only (-1)-curves in $S'_{\overline{K}}$ intersecting $E_{0,\overline{K}}$. The union $\ell_1\cup\ell_2\cup\ell_3$ is thus defined over K, and since these curves are disjoint in $S'_{\overline{K}}$, they can be simultaneously contracted, giving rise to a birational morphism $h: S'\to S''$ defined over K onto a smooth del Pezzo surface S'' of Picard number one and degree 8, hence again a nontrivial K-form of $\mathbb{P}^1_K\times\mathbb{P}^1_K$, containing at least a K-rational point q supported on the image E''_0 of E_0. The image of $E''_{0,\overline{K}}$ in $S''_{\overline{K}}$ is an irreducible curve of type $(1,1)$ in the divisor class group $\mathrm{Cl}(S''_{\overline{K}})\simeq\mathbb{Z}^2$ of $S''_{\overline{K}}$. The union of the fibers of the first and second projection passing through q in $S''_{\overline{K}}$ is another curve C of type $(1,1)$ defined over K. So $E''_{0,\overline{K}}$ and C generate a pencil $\varphi: S''\dashrightarrow\mathbb{P}^1_K$ defined over K, having q as a unique proper base point. The latter restricts to a trivial $\mathbb{A}^1_K$-bundle $S''\setminus(E''_0\cup C)\to\mathbb{P}^1_K\setminus(\varphi(E''_0)\cup\varphi(C))\simeq\mathbb{A}^1_{*,K}$ over $\mathbb{A}^1_{*,K}\simeq\mathrm{Spec}(K[t^{\pm1}])$. By construction, $g\circ h^{-1}$ induces an isomorphism between $S''\setminus(E''_0\cup C)$ and its image $U\subset S$, and $\varphi\circ h\circ g^{-1}: U\to\mathbb{A}^1_{*,K}$ is isomorphic to trivial $\mathbb{A}^1_K$-bundle $\mathbb{A}^1_{*,K}\times\mathbb{A}^1_K$ over $\mathbb{A}^1_{*,K}$.

(iv) If $d=5$ then $S_{\overline{K}}$ is isomorphic to the blow-up $\tau: S_{\overline{K}}\to\mathbb{P}^2_{\overline{K}}$ of $\mathbb{P}^2_{\overline{K}}$ at four points $x_1,\ldots,x_4$ in general position. The union D of all the (-1)-curves in $S_{\overline{K}}$ is defined over K, and consists of the exceptional divisors of τ and the proper transforms of the lines in $\mathbb{P}^2_{\overline{K}}$ passing through x_i and x_j, $1\le i<j\le 4$. Since $\rho(S)=1$ it follows from [13, 29.4.4 (v)] that a K-rational point p of S is necessarily supported outside of D. Let $g: S'\to S$ be the blow-up of such a K-rational point p, with exceptional divisor $E_0\simeq\mathbb{P}^1_K$. Then S' is a del Pezzo surface of degree 4, and the union D' of all (-1)-curves in $S'_{\overline{K}}$ consists of $E_{0,\overline{K}}$, the proper transform of D, the proper transforms ℓ_i by $\tau\circ g_{\overline{K}}$ of the lines in $\mathbb{P}^2_{\overline{K}}$ passing through $\tau(p)$ and $x_1,\ldots,x_4$, and the proper transform C of the unique smooth conic in $\mathbb{P}^2_{\overline{K}}$ passing through $\tau(p)$, $x_1,\ldots,x_4$. Since C, $\ell_1,\ldots,\ell_4$ are the only (-1)-curves intersecting the proper transform of $E_{0,\overline{K}}$, their union is defined over K, and since they are also disjoint, they can therefore be simultaneously contracted. This yields a birational morphism $h: S'\to S''$ defined over K onto a smooth del Pezzo surface of degree 9, containing a K-rational point supported on the image E''_0 of E_0. So $S''\simeq\mathbb{P}^2_K$ in which E''_0 is a smooth conic, with a K-rational point q. Then E''_0 and twice is tangent line $T_q(E''_0)\simeq\mathbb{P}^1_K$ at p generate a pencil $\varphi: S''\dashrightarrow\mathbb{P}^1_K$ defined over K, having q as a unique proper base point and whose restriction to $\mathbb{P}^2_K\setminus(E''_0\cup T_q(E''_0))$ is a trivial $\mathbb{A}^1_K$-bundle over $\mathbb{P}^1_K\setminus(\varphi(E''_0)\cup\varphi(T_q(E''_0)))\simeq\mathbb{A}^1_{*,K}$. By construction, $g\circ h^{-1}$ induces an isomorphism between $S''\setminus(E''_0\cup T_q(E''_0))$ and its image $U\subset S$, and $\varphi\circ h\circ g^{-1}: U\to\mathbb{A}^1_{*,K}$ is isomorphic to trivial $\mathbb{A}^1_K$-bundle $\mathbb{A}^1_{*,K}\times\mathbb{A}^1_K$ over $\mathbb{A}^1_{*,K}$. □

1.4. **Complement: $\mathbb{A}^2$-cylinders in del Pezzo surfaces of Picard number one.** It follows from the proof of Proposition 12 that every smooth del Pezzo surface S of degree $d\ge 8$ with a K-rational point actually contains $\mathbb{A}^2_K$ as an open subset. Here we show in contrast that this is no longer the case for del Pezzo surfaces of degree 5 and 6 with Picard number one, namely:

Proposition 13. *A smooth del Pezzo surface S defined over a field K of characteristic zero 0, with Picard number one and of degree $d=5$ or 6 does not contain $\mathbb{A}^2_K$ as an open subset.*

Proof. We proceed by contradiction, assuming that S contains an open subset U isomorphic to $\mathbb{A}^2_K$. Since $\mathrm{Pic}(S)$ is generated by $-K_S$ and $\mathrm{Pic}(U)$ is trivial, $B=S\setminus U$ is a reduced, irreducible effective anti-canonical divisor on S, defined over K. Furthermore, the irreducible components of $B_{\overline{K}}$ must form a basis of the Picard group of $S_{\overline{K}}$ on which the Galois group $\mathrm{Gal}(\overline{K}/K)$ acts transitively.

(i) If $d=6$, then $\mathrm{Pic}(S_{\overline{K}}) \simeq \mathbb{Z}^4$ and we can write $B_{\overline{K}} = B_1+B_2+B_3+B_4$, where $B_1,\ldots,B_4$ are irreducible curves forming a basis of $\mathrm{Pic}(S_{\overline{K}})$ on which $\mathrm{Gal}(\overline{K}/K)$ acts transitively. The equality

$$6 = (-K_{S_{\overline{K}}}^2) = B_{\overline{K}}^2 = -\sum_{i=1}^{4} K_{S_{\overline{K}}} \cdot B_i$$

implies that $K_{S_{\overline{K}}} \cdot B_i = -1$ for at least one i. The corresponding curve is thus a (-1)-curve, and so, the B_i are all (-1)-curves. This would imply in turn that $4=6$, which is absurd.

(ii) If $d=5$, then $\mathrm{Pic}(S_{\overline{K}}) \simeq \mathbb{Z}^5$ and similarly as above, we can write $B_{\overline{K}} = \sum_{i=1}^{5} B_i$, where $B_1,\ldots,B_5$ are irreducible curves forming a basis of $\mathrm{Pic}(S_{\overline{K}})$ on which $\mathrm{Gal}(\overline{K}/K)$ acts transitively. The equality

$$5 = (-K_{S_{\overline{K}}}^2) = B_{\overline{K}}^2 = -\sum_{i=1}^{5} K_{S_{\overline{K}}} \cdot B_i$$

implies again that all the B_i are (-1)-curves. Since $S_{\overline{K}} \setminus B_{\overline{K}} \simeq \mathbb{A}^2_{\overline{K}}$, it follows from [14] that the support of the total transform of $B_{\overline{K}}$ in a minimal log-resolution of the pair $(S_{\overline{K}}, B_{\overline{K}})$ is a tree of rational curves. Therefore, the support of $B_{\overline{K}}$ is connected and does contain any cycle, and since $\mathrm{Gal}(\overline{K}/K)$ acts transitively on its irreducible components, we conclude that the curves B_i intersect each others in a unique common point. Letting $c = \min_{1\le i<j\le 5}\{(B_i \cdot B_j)\} \geq 1$ be the minimum of the intersection number between two distinct components of $B_{\overline{K}}$, we obtain that

$$5 = (-K_{S_{\overline{K}}}^2) = (B_1+\cdots+B_5)^2 \geq -5+20c,$$

which is absurd as $c \geq 1$. □

Corollary 14. *A complex del Pezzo fibration $\pi : V \to W$ admits a vertical $\mathbb{A}^2$-cylinder if and only if $\deg(V/W) = 8$ or 9 and $\pi : V \to W$ has a rational section.*

2. Examples of threefold del Pezzo fibrations containing twisted cylinders

In what follows we first briefly review the general setup for the construction of projective completions of $\mathbb{A}^3_{\mathbb{C}}$ into total spaces of del Pezzo fibrations $\pi : V \to \mathbb{P}^1$ established in [4]. Then we give a detailed construction for the specific case $d=4$, which was announced without proof in [4], thus completing the proof of Theorem 2.

2.1. General setup and existence results in degree ≤ 3.

2.1.1. *Step 1: Pencils of del Pezzo surfaces.* We begin with a smooth del Pezzo surface S of degree $d \leq 3$ anti-canonically embedded as a hypersurface of degree e in a weighted projective space $\mathbb{P} = \mathrm{Proj}(\mathbb{C}[x,y,z,w])$. So $\mathbb{P}$ is equal $\mathbb{P}^3$, $\mathbb{P}(1,1,1,2)$ and $\mathbb{P}(1,1,2,3)$ and e is equal to 3, 4 or 6 according as $d=3$, 2 and 1. Given a hyperplane $H \in |\mathcal{O}_{\mathbb{P}}(1)|$, the open subset $U = \mathbb{P} \setminus H$ is isomorphic to $\mathbb{A}^3_{\mathbb{C}}$. We let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ the pencil generated by S and eH and we denote by $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ the corresponding rational map. We let $\infty = \overline{f}_*(H) \in \mathbb{P}^1$.

2.1.2. *Step 2: Good resolutions.* Next we take a *good resolution* of the indeterminacies of $\overline{f}$, that is, a triple $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ consisting of a projective threefold $\tilde{\mathbb{P}}$, a birational morphism $\sigma : \tilde{\mathbb{P}} \to \mathbb{P}$ and a morphism $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ satisfying the following properties:

a) The diagram

$$\begin{tikzcd} \tilde{\mathbb{P}} \arrow[r, "\sigma"] \arrow[d, "\tilde{f}"'] & \mathbb{P} \arrow[d, dashed, "\overline{f}"] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \end{tikzcd}$$

commutes.

b) $\tilde{\mathbb{P}}$ has at most $\mathbb{Q}$-factorial terminal singularities and is smooth outside $\tilde{f}^{-1}(\infty)$.

c) $\sigma : \tilde{\mathbb{P}} \to \mathbb{P}$ is a sequence of blow-ups whose successive centers lie above the base locus of $\mathcal{L}$, inducing an isomorphism $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \stackrel{\sim}{\to} \mathbb{P} \setminus H$, and whose restriction to every closed fiber of $\tilde{f}$ except $\tilde{f}^{-1}(\infty)$ is an isomorphism onto its image.

Such a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of $f : \mathbb{P} \dashrightarrow \mathbb{P}^1$ always exists and can be for instance obtained by first taking the blow-up $\tau : X \to \mathbb{P}$ of the scheme-theoretic base locus of $\mathcal{L}$ and then any resolution $\tau_1 : \tilde{\mathbb{P}} \to X$ of the singularities of X. In this case, the triple $(\tilde{\mathbb{P}}, \tau \circ \tau_1, \overline{f} \circ \pi \circ \tau_1)$ is a good resolution of $\overline{f}$ for which $\tilde{\mathbb{P}}$ is even smooth.

The definition implies that the generic fiber $\tilde{\mathbb{P}}_\eta$ of $\tilde{f}$ is a smooth del Pezzo surface of degree d defined over the field of rational functions K of $\mathbb{P}^1$. The irreducible divisors in the exceptional locus $\mathrm{Exc}(\sigma)$ of σ that are vertical for $\tilde{f}$, i.e. contained in closed fibers of $\tilde{f}$, are all contained in $\tilde{f}^{-1}(\infty)$. On the other hand, $\mathrm{Exc}(\sigma)$ contains exactly as many irreducible horizontal divisors as there are irreducible components in $H \cap S$, and $\sigma^{-1}(H)$ intersects $\tilde{\mathbb{P}}_\eta$ along the curve $D_\eta \simeq (H \cap S) \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(K)$ which is an anti-canonical divisor on $\tilde{\mathbb{P}}_\eta$ with the same number of irreducible components as $H \cap S$. Note also that by assumption $\tilde{U} = \tilde{\mathbb{P}} \setminus \sigma^{-1}(H)$ is again isomorphic to $\mathbb{A}^3_{\mathbb{C}}$.

2.1.3. *Step 3: Relative MMP.* The next step consists in running a MMP $\varphi : \tilde{\mathbb{P}}_0 = \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}' = \tilde{\mathbb{P}}_n$ relative to the morphism $\tilde{f}_0 = \tilde{f} : \tilde{\mathbb{P}}_0 \to \mathbb{P}^1$. Recall [11, 3.31] that such a relative MMP consists of a finite sequence $\varphi = \varphi_n \circ \cdots \circ \varphi_1$ of birational maps

$$\begin{array}{ccc} \tilde{\mathbb{P}}_{k-1} & \stackrel{\varphi_k}{\dashrightarrow} & \tilde{\mathbb{P}}_k \\ \tilde{f}_{k-1} \downarrow & & \downarrow \tilde{f}_k \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array} \qquad k = 1, \ldots, n,$$

where each φ_k is associated to an extremal ray R_{k-1} of the closure $\overline{NE}(\tilde{\mathbb{P}}_{k-1}/\mathbb{P}^1)$ of the relative cone of curves of $\tilde{\mathbb{P}}_{k-1}$ over $\mathbb{P}^1$. Each of these birational maps φ_k is either a divisorial contraction or a flip whose flipping and flipped curves are contained in the fibers of $\tilde{f}_{k-1}$ and $\tilde{f}_k$ respectively. The following crucial result established in [4] guarantees that every such relative MMP terminates with a projective threefold $\tilde{\mathbb{P}}'$ with at most $\mathbb{Q}$-factorial terminal singularities containing $\mathbb{A}^3_{\mathbb{C}}$ as an open subset.

Proposition 15. *Let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ be as above and let $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ be any good resolution of the corresponding rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$. Then every MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ relative to $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ restricts to an isomorphism $\mathbb{A}^3_{\mathbb{C}} \simeq \tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \stackrel{\sim}{\to} \tilde{\mathbb{P}}' \setminus \varphi_*(\sigma^{-1}(H))$.*

In particular, the restriction of every MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ relative to $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ between the generic fibers $\tilde{\mathbb{P}}_\eta$ and $\tilde{\mathbb{P}}'_\eta$ of $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ and $\tilde{f}' : \tilde{\mathbb{P}} \to \mathbb{P}^1$ respectively is either an isomorphism onto its image, or the contraction of a finite sequence of successive (-1)-curves among the irreducible components of $D_\eta = \tilde{\mathbb{P}}_\eta \cap \sigma^{-1}(H)$. It was shown in addition in [4] that for every $k = 1, \ldots, n$, the restriction of φ_k to every closed fiber of $\tilde{f}_{k-1} : \tilde{\mathbb{P}}_{k-1} \to \mathbb{P}^1$ distinct from $\tilde{f}_{k-1}^{-1}(\infty)$ is either an isomorphism onto the corresponding fiber of $\tilde{f}_k : \tilde{\mathbb{P}}_k \to \mathbb{P}^1$ or the contraction of finitely many disjoint (-1)-curves. In particular, in the case where $\varphi_k : \tilde{\mathbb{P}}_{k-1} \dashrightarrow \tilde{\mathbb{P}}_k$ is a flip, then all its flipping and flipped curves are contained in $\tilde{f}_{k-1}^{-1}(\infty)$ and $\tilde{f}_k^{-1}(\infty)$ respectively.

2.1.4. *Step 4: Determination of the possible outputs.* Since a general member of a pencil $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ is a rational surface, the output $\tilde{\mathbb{P}}'$ of a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ ran from a good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ of the corresponding rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ is a Mori fiber space $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$. So $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is either a del Pezzo fibration with relative Picard number one, whose degree is fully determined by that of the initial del Pezzo surface $S \subset \mathbb{P}$ and the number of (-1)-curves in the generic fiber of $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ contracted by φ, or it factors through a Mori conic bundle $\xi : \tilde{\mathbb{P}}' \to X$ over a certain normal projective surface $q : X \to \mathbb{P}^1$. The following theorem established in [4] shows that except maybe in the case where $d = 3$ and $H \cap S$ consists of two irreducible components, the structure of $\tilde{\mathbb{P}}'$ depends only on the base locus of $\mathcal{L}$. In particular, it depends neither on the chosen good resolution $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ nor on a particular choice of a relative MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$.

Theorem 16. *Let $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$ be the pencil generated by a smooth del Pezzo surface $S \subset \mathbb{P}$ of degree $d \in \{1, 2, 3\}$ and eH for some $H \in |\mathcal{O}_{\mathbb{P}}(1)|$, let $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ be a good resolution of the corresponding rational map $\overline{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$, and let $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ be a relative MMP. Then the following hold:*

a) If $H \cap S$ is irreducible, then $\tilde{f}' : \tilde{\mathbb{P}}' \to \mathbb{P}^1$ is a del Pezzo fibration of degree d.

b) If $d=2$ and $H\cap S$ is reducible, then $\tilde{f}':\tilde{\mathbb{P}}'\to\mathbb{P}^1$ is del Pezzo fibration of degree $d+1=3$.

c) If $H\cap S$ has three irreducible components, then $\tilde{f}':\tilde{\mathbb{P}}'\to\mathbb{P}^1$ factors through a Mori conic bundle $\xi:\tilde{\mathbb{P}}'\to X$ over a normal projective surface $q:X\to\mathbb{P}^1$.

As a consequence, we obtain the following existence result:

Corollary 17. *Let $\mathcal{L}\subset|\mathcal{O}_{\mathbb{P}}(e)|$ be the pencil generated by a smooth del Pezzo surface $S\subset\mathbb{P}$ of degree $d\in\{1,2,3\}$ and eH for some $H\in|\mathcal{O}_{\mathbb{P}}(1)|$ such that $H\cap S$ is irreducible. Then for every good resolution $(\tilde{\mathbb{P}},\sigma,\tilde{f})$ of the corresponding rational map $\overline{f}:\mathbb{P}\dashrightarrow\mathbb{P}^1$ and every MMP $\varphi:\tilde{\mathbb{P}}\dashrightarrow\tilde{\mathbb{P}}'$ relative to $\tilde{f}:\tilde{\mathbb{P}}\to\mathbb{P}^1$, the output $\tilde{f}':\tilde{\mathbb{P}}'\to\mathbb{P}^1$ is a del Pezzo fibration of degree d whose total space $\tilde{\mathbb{P}}'$ contains $\mathbb{A}^3_{\mathbb{C}}$ as a Zariski open subset.*

2.2. **Existence results in degree** 4. In Theorem 16, the remaining case where S is smooth cubic in $\mathbb{P}^3$ and $H\cap S$ consists of two irreducible components, namely a line L and smooth conic C intersecting each others twice, is more complicated. Here given a good resolution $(\tilde{\mathbb{P}},\sigma,\tilde{f})$ of the rational map $\overline{f}:\mathbb{P}=\mathbb{P}^3\dashrightarrow\mathbb{P}^1$, the intersection of $\sigma^{-1}(H)$ with the generic fiber $\tilde{\mathbb{P}}_\eta$ of $\tilde{f}:\tilde{\mathbb{P}}\to\mathbb{P}^1$ is a reduced anti-canonical divisor whose support consists of the union of a (-1)-curve L_η and of a 0-curve C_η both defined over the function field K of $\mathbb{P}^1$. By Proposition 15, the only horizontal divisors contracted by a relative MMP $\varphi:\tilde{\mathbb{P}}\dashrightarrow\tilde{\mathbb{P}}'$ are irreducible components of $\sigma^{-1}(H)$. In this case, it follows that φ can contract at most the irreducible component of $\sigma^{-1}(H)$ intersecting $\tilde{\mathbb{P}}_\eta$ along L_η. Indeed, if L_η is contracted at the certain step $\varphi_k:\tilde{\mathbb{P}}_{k-1}\dashrightarrow\tilde{\mathbb{P}}_k$ then the image of C_η in the generic fiber of $\tilde{f}_k:\tilde{\mathbb{P}}_k\to\mathbb{P}^1$ is a singular curve with positive self-intersection which therefore cannot be contracted at any further step $\varphi_{k'}$, $k'\geq k+1$, of φ. If L_η is contracted, then the generic fiber of the output $\tilde{f}':\tilde{\mathbb{P}}'\to\mathbb{P}^1$ of φ is a smooth del Pezzo surface of degree 4, and it was established in [4, Proposition 11] that in this case, $\tilde{f}':\tilde{\mathbb{P}}'\to\mathbb{P}^1$ is in fact a del Pezzo fibration of degree 4. The following result was announced without proof in [4]:

Proposition 18. *Let $S\subset\mathbb{P}^3$ be a smooth cubic surface, let $H\in|\mathcal{O}_{\mathbb{P}^3}(1)|$ be a hyperplane intersecting S along the union of a line and smooth conic, let $\mathcal{L}\subset|\mathcal{O}_{\mathbb{P}^3}(3)|$ be the pencil generated by S and $3H$ and let $\overline{f}:\mathbb{P}^3\dashrightarrow\mathbb{P}^1$ be the corresponding rational map. Then there exists a good resolution $(\tilde{\mathbb{P}},\sigma,\tilde{f})$ and a MMP $\varphi:\tilde{\mathbb{P}}\dashrightarrow\tilde{\mathbb{P}}'$ relative to $\tilde{f}:\tilde{\mathbb{P}}\to\mathbb{P}^1$ whose output is a del Pezzo fibration $\tilde{f}':\tilde{\mathbb{P}}'\to\mathbb{P}^1$ of degree* 4.

In particular, $\tilde{f}':\tilde{\mathbb{P}}'\to\mathbb{P}^1$ is a del Pezzo fibration of degree 4 whose total space $\tilde{\mathbb{P}}'$ contains $\mathbb{A}^3_{\mathbb{C}}$ as a Zariski open subset.

The rest of this subsection is devoted to the proof of this proposition. In view of the above discussion, it is enough to construct a particular good resolution $\sigma:\tilde{\mathbb{P}}\to\mathbb{P}^3$ of $\overline{f}:\mathbb{P}^3\dashrightarrow\mathbb{P}^1$ for which there exists a MMP $\varphi:\tilde{\mathbb{P}}\dashrightarrow\tilde{\mathbb{P}}'$ relative to $\tilde{f}:\tilde{\mathbb{P}}\to\mathbb{P}^1$ whose first step consists of the contraction of the irreducible component of $\sigma^{-1}(H)$ intersecting the generic fiber of $\tilde{f}$ along the (-1)-curve L_η.

2.2.1. *Construction of a particular good resolution $\sigma:\tilde{\mathbb{P}}\to\mathbb{P}^3$.* Let again $\mathcal{L}\subset|\mathcal{O}_{\mathbb{P}^3}(3)|$ be the pencil generated by a smooth cubic surface $S\subset\mathbb{P}^3$ and $3H$, where $H\in|\mathcal{O}_{\mathbb{P}^3}(1)|$ is a hyperplane intersecting S along the union of a line L and smooth conic C. In what follows, starting from $\mathbb{P}_0=\mathbb{P}^3$, we construct a resolution of the base locus of $\mathcal{L}$ consisting of a sequence of blow-ups

$$\sigma=\sigma_1\circ\cdots\circ\sigma_6:\tilde{\mathbb{P}}=\mathbb{P}_6\xrightarrow{\sigma_6}\mathbb{P}_5\xrightarrow{\sigma_5}\cdots\xrightarrow{\sigma_2}\mathbb{P}_1\xrightarrow{\sigma_1}\mathbb{P}_0=\mathbb{P}^3$$

along successive smooth centers. To fix the notation, we let S_i and H_i the proper transforms of S and H on $\mathbb{P}_i$, respectively. Similarly, we denote by $\mathcal{L}_i$ the proper transform on $\mathbb{P}_i$ of the pencil $\mathcal{L}$. We denote by E_i the exceptional divisor of σ_i, and we use the same notation to denote its proper transform on any other $\mathbb{P}_j$. The base locus $\mathrm{Bs}\mathcal{L}$ of $\mathcal{L}$ is supported by the union of C and L. We proceed in two steps:

Step 1) First we let $\sigma_1:\mathbb{P}_1\to\mathbb{P}_0$ be the blow-up of $\mathbb{P}_0$ with center at C. Since C is a Cartier divisor on S, σ_1 restricts to an isomorphism between S_1 and S, in particular S_1 is again smooth and $E_1\cap S_1$ is a smooth conic C_1 which is mapped isomorphically onto C by the restriction of σ_1. We have $S_1\sim 3H_1+2E_1$, $\mathcal{L}_1$ is spanned by S_1 and $3H_1+2E_1$, and $\mathrm{Bs}\mathcal{L}_1$ is supported by the union of C_1 with the proper transform L_1 of L.

Next we let $\sigma_2:\mathbb{P}_2\to\mathbb{P}_1$ be the blow-up of $\mathbb{P}_1$ with center at C_1. Similarly as in the previous case, σ_2 restricts to an isomorphism between S_2 and S_1, $E_2\cap S_2$ is a smooth conic C_2 which is mapped isomorphically onto C_1 by σ_2. Furthermore, $S_2\sim 3H_2+2E_1+E_2$, the pencil $\mathcal{L}_2$ is spanned by S_2 and $3H_2+2E_1+E_2$, and its base locus is supported on the union of C_2 with the proper transform L_2 of L_1.

Then we let $\sigma_3 : \mathbb{P}_3 \to \mathbb{P}_2$ be the blow-up of $\mathbb{P}_2$ with center at C_2. Again, σ_3 restricts to an isomorphism between S_3 and S_2. We have $S_3 \sim 3H_3 + 2E_1 + E_2$, the pencil $\mathcal{L}_3$ is spanned by S_3 and $3H_3 + 2E_1 + E_2$, and $\mathrm{Bs}\mathcal{L}_3$ is supported by the line L_3 which is the proper transform of L by the isomorphism $S_3 \tilde{\to} S$ induced by $\sigma_1 \circ \sigma_2 \circ \sigma_3$.

Step 2). We let $\sigma_4 : \mathbb{P}_4 \to \mathbb{P}_3$ be the blow-up of $\mathbb{P}_3$ with center at L_3. A similar argument as above implies that $S_4 \sim 3H_4 + 2E_1 + E_2 + 2E_4$, that $\mathcal{L}_4$ is spanned by S_4 and $3H_4 + 2E_1 + E_2 + 2E_4$ and that its base locus is supported on the line $L_4 = E_4 \cap S_4$ which is mapped isomorphically onto L_3 by the restriction of σ_4 to S_4.

Then we let $\sigma_5 : \mathbb{P}_5 \to \mathbb{P}_4$ be the blow-up of $\mathbb{P}_4$ with center at L_4. The pencil $\mathcal{L}_5$ is generated by S_5 and $3H_5 + 2E_1 + E_2 + 2E_4 + E_5$, and its base locus is equal to the line $L_5 = E_5 \cap S_5$.

Finally, we let $\sigma_6 : \mathbb{P}_6 \to \mathbb{P}_5$ be the blow-up of $\mathbb{P}_5$ with center at L_5. By construction, $\mathbb{P}_6$ is a smooth threefold on which the proper transform $\mathcal{L}_6$ of $\mathcal{L}$ is the base point free pencil generated by $S_6 \simeq S$ and $3H_6 + 2E_1 + E_2 + 2E_4 + E_5$.

Summing up, we obtained:

Lemma 19. *With the notation above, the following hold:*

a) The birational morphism $\sigma = \sigma_1 \circ \cdots \cdot \sigma_6 : \tilde{\mathbb{P}} = \mathbb{P}_6 \to \mathbb{P}^3$ is a good resolution of $\overline{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$.

b) The divisors E_3 and E_6 are the horizontal irreducible components of $\sigma^{-1}(H)$, and they intersect the generic fiber of $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ along a 0-curve C_η and (-1)-curve L_η respectively.

2.2.2. *Existence of a suitable relative MMP.* To complete the proof of Proposition 18, it remains to check that E_6 can be contracted at a step of a MMP $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ relative to $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$. For this, it suffices to show that the restriction $\sigma_6 \mid_{E_6} : E_6 \to L_5$ is isomorphic to the trivial $\mathbb{P}^1$-bundle $\mathrm{pr}_2 : E_6 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to L_5 \simeq \mathbb{P}^1$ and that the class of a fiber of the first projection generates an extremal ray R of the closure $\overline{NE}(\tilde{\mathbb{P}}/\mathbb{P}^1)$ of the relative cone of curves of $\tilde{\mathbb{P}}$ over $\mathbb{P}^1$. Indeed, if so, the contraction $\varphi : \tilde{\mathbb{P}} \to \tilde{\mathbb{P}}_1$ associated to this extremal ray is the first step of a MMP relative to the morphism $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}^1$ consisting of the divisorial contraction of E_6 onto a smooth curve isomorphic to $\mathbb{P}^1$. The existence of R is an immediate consequence of the following lemma which completes the proof of Proposition 18.

Lemma 20. *The pair $(E_6, \mathcal{N}_{E_6/\tilde{\mathbb{P}}})$ is isomorphic to $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1))$.*

Proof. The normal bundle $\mathcal{N}_{S/\mathbb{P}^3}$ of S in $\mathbb{P}^3$ is isomorphic to $\mathcal{O}_S(3(C+L))$, and by construction of the resolution $\sigma : \tilde{\mathbb{P}} = \mathbb{P}_6 \to \mathbb{P}^3$, it follows that $\mathcal{N}_{S_i/\mathbb{P}_i} \simeq \mathcal{O}_{S_i}((3-i)C_i + 3L_i)$, $i = 1,2,3$, $\mathcal{N}_{S_4/\mathbb{P}_4} = \mathcal{O}_{S_4}(2L_4)$ and $\mathcal{N}_{S_5/\mathbb{P}_5} = \mathcal{O}_{S_5}(L_5)$. Since L_5 is a line in the cubic surface S_5, we have $\mathcal{N}_{L_5/S_5} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{N}_{S_5/\mathbb{P}_5} \mid_{L_5} \simeq \mathcal{O}_{\mathbb{P}^1}(L_5^2) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, where L_5^2 denotes the self-intersection of L_5 on the surface S_5. It then follows from the exact sequence

$$0 \to \mathcal{N}_{L_5/S_5} \to \mathcal{N}_{L_5/\mathbb{P}_5} \to \mathcal{N}_{S_5/\mathbb{P}_5} \mid_{L_5} \to 0$$

and the vanishing of $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$, that $\mathcal{N}_{L_5/P_5} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Thus $E_6 \simeq \mathbb{P}(\mathcal{N}_{L_5/\mathbb{P}_5})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Since $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ is of type $(-2,-2)$ in the Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$, it is enough to show that $K_{E_6} = 2E_6 \mid_{E_6}$. Let L_0 be a fiber of the restriction of $\sigma_6 \mid_{E_6} : E_6 \to L_5$ of σ_6. Since $K_{\mathbb{P}_6} = \sigma_6^* K_{\mathbb{P}_5} + E_6$, it follows from the adjunction formula that

$$K_{E_6} = (K_{\mathbb{P}_6} + E_6) \mid_{E_6} = (K_{\mathbb{P}_5} \cdot L_5) L_0 + 2E_6 \mid_{E_6} .$$

On the other hand, since L_5 is a (-1)-curve on S_5 and $\mathcal{N}_{S_5/\mathbb{P}_5} \mid_{L_5} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, we have

$$1 = -K_{S_5} \cdot L_5 = (-(K_{\mathbb{P}_5} + S_5) \mid_{S_5} \cdot L_5) = -K_{\mathbb{P}_5} \cdot L_5 + 1.$$

Thus $K_{\mathbb{P}_5} \cdot L_5 = 0$, and hence $K_{E_6} = 2E_6 \mid_{E_6}$ as desired. □

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